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# $P$-points in $\mathbb{N}^*$ and the spaces of continuous functions

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**Abstract**

It is proved that a point $\phi$ from the Čech–Stone remainder $\mathbb{N}^*$ is a $P$-point iff $C_p(\mathbb{N}_\phi)$ is a hereditary Baire space, where $\mathbb{N}_\phi = \mathbb{N} \cup \{\phi\}$. Some characterizations of $P$-points in terms of games played in $\mathbb{N}_\phi$ and $C_p(\mathbb{N}_\phi)$ are also given. © 1998 Published by Elsevier Science B.V.

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## Introduction

This paper answers the question posed to the authors by Cauty: *Is it possible to embed the space $\mathbb{Q}$ of all rational numbers with usual topology into the space $C_p(\mathbb{N}_\phi)$ as a closed subspace*? There $C_p(\cdot)$ denotes a space of all continuous functions with pointwise topology and $\mathbb{N}_\phi$ denotes a subspace $\mathbb{N} \cup \{\phi\}$ of the Čech–Stone compactification $\beta\mathbb{N}$, where $\phi \in \mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. Due to the classical Hurewicz theorem [3] (see also [1,2]), a separable metric space is a hereditary Baire space if, and only if, it does not contain the space $\mathbb{Q}$ as a closed subspace. Therefore, the above question of Cauty concerns a characterization of the hereditary Baire property for the space $C_p(\mathbb{N}_\phi)$. It was proved by Lutzer and McCoy [5] that $C_p(\mathbb{N}_\phi)$ is a Baire space for any ultrafilter $\phi$. What is a difference between the Baire property and its hereditary analogue in the case of spaces $C_p(\mathbb{N}_\phi)$? The main result of this paper shows that this function space usually does not possess the hereditary variant of the Baire property, so both above properties are rather different. More exactly, the space $C_p(\mathbb{N}_\phi)$ has the hereditary Baire property only in the

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case that $\phi$ is a $P$-point in $\mathbb{N}^*$. By the well-known result of Shelah, the existence of a $P$-point in $\mathbb{N}^*$ is possible only in special models of ZFC.

In our proofs we follow some ideas due to Debs [2] and we use his characterization of hereditary Baire space in terms of a winning strategy for the Choquet game $\Gamma$ in the space $C_p(\mathbb{N}_\phi)$. We discover the game $\Delta$ for the space $\mathbb{N}_\phi$ which is completely *dual* to the game $\Gamma$ for the space $C_p(\mathbb{N}_\phi)$ and this allows us to prove our main theorem. We suppose that this idea of duality under the $C_p$-functor between games could find a further development. *What other pairs of games are in duality*?

Our result allows us to formulate the problem of characterization of the hereditary Baire property for spaces $C_p(X)$. Observe that an analogous problem for the Baire property of function space $C_p(X)$ was solved independently by Pytkeev [4] and Tkachuk [6].

Our terminology and definitions are standard. A space is called Baire if every intersection of a countable family of open dense subsets is dense in it. A space is hereditary Baire if its each closed subset is Baire. The game $\Gamma$ of two players I and II is called a *Choquet game* (cf. [2]) in space $X$ if player I takes an open set $U_n$ with a point $x_n \in U_n$ and player II takes an open set $V_n$ in $X$ such that $U_{n+1} \subset V_n \subset U_n$ and $x_n \in V_n$ for every $n = 0, 1, 2, \ldots$. Player I wins the play if $\bigcap_{n=0}^\infty U_n = \bigcap_{n=0}^\infty V_n = \emptyset$, player II wins otherwise. As we remark above, in this paper we introduce the new game $\Delta$ which is played in the space $\mathbb{N}_\phi$ and is dual to the Choquet game $\Gamma$ in the space $C_p(\mathbb{N}_\phi)$. Its definition is as follows. Player I begins and he takes sets $S_n$ in $\mathbb{N}$ with $S_n \notin \phi$, and player II replies by choosing finite sets $T_n$ in $\mathbb{N}$ for every $n = 0, 1, 2, \ldots$ in such a manner that the whole system of sets $S_n$ and $T_n$ is pairwise disjoint. Remark that both players are allowed to choose empty sets as some of their moves. We define that player II wins the play if $\bigcup_{n=0}^\infty T_n \in \phi$.

By the symbol $\mathcal{F}$, we will denote the subspace of the product $\mathbb{R}^\mathbb{N}$ consisting of all functions $f : \mathbb{N} \to \mathbb{R}$ with a finite support $\operatorname{supp} f = \{n \in \mathbb{N}\colon\ f(n) \neq 0\}$.

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## Main result

The following statement is well known and appears here for the sake of completeness.

**Lemma 1.** *For any ultrafilter $\phi$ on $\mathbb{N}$, the space $C_p(\mathbb{N}_\phi)$ is linearly homeomorphic to its subspace*

$$C_p^0(\mathbb{N}_\phi) = \{f \in C_p(\mathbb{N}_\phi)\colon\ \lim_\phi f(n) = f(\{\phi\}) = 0.\}$$

**Proof.** The mapping $u(f) = (f(\{\phi\}), f(\cdot) - f(\{\phi\}))$ is a linear homeomorphism of $C_p(\mathbb{N}_\phi)$ onto $\mathbb{R} \times C_p^0(\mathbb{N}_\phi)$. There is an infinite subset $A \subset \mathbb{N}$ with $A \notin \phi$, hence $C_p^0(\mathbb{N}_\phi) = C_p^0(\mathbb{N}_\phi \backslash A) \times \mathbb{R}^A$ and the result follows from an evident linear homeomorphism between spaces $\mathbb{R} \times \mathbb{R}^A$ and $\mathbb{R}^A$. $\square$

**Lemma 2.** *Let $\mathcal{B}$ be a base of a topological space $X$. Denote by $\Gamma_{\mathcal{B}}$ the Choquet game in $X$ in which both players are restricted to choose moves from $\mathcal{B}$.*

*Then player I has a winning strategy in the game $\Gamma$ iff he has a winning strategy in the game $\Gamma_{\mathcal{B}}$.*

**Proof.** Let $\sigma$ be a strategy for player I in the game $\Gamma$ with the beginning step $\sigma(\emptyset) = \{U_0, x_0\}$. Introduce a strategy $\sigma'$ in the game $\Gamma_{\mathcal{B}}$ in the following way. Fix some $U_0' \in \mathcal{B}$ such that $x_0 \in U_0' \subset U_0$. Put $\sigma'(\emptyset) = \{U_0', x_0\}$. When player II takes some $V_0' \in \mathcal{B}$ with $x_0 \in V_0' \subset U_0'$ then player I finds the pair $(U_1, x_1) = \sigma(U_0, x_0, V_0')$ and some $U_1' \in \mathcal{B}$ with $x_1 \in U_1' \subset U_1$. Put $(U_1', x_1) = \sigma'(U_0', x_0, V_0')$ and so on. This gives us the decreasing sequence of open sets $U_0 \supset U_0' \supset V_0' \supset U_1 \supset U_1' \supset V_1' \supset \cdots$. When the play associated with the choice of sets $U_0, U_1, \ldots$ (according to the strategy for player I in $\Gamma$) will be finished, then we get $\bigcap U_n = \emptyset$ by the assumption. Therefore $\bigcap U_n' = \bigcap V_n' = \emptyset$ and player I wins in the 'prime' play $\Gamma_{\mathcal{B}}$. This proves the 'if' part of the lemma.

Suppose now that player I has a winning strategy $\sigma'$ in the game $\Gamma_{\mathcal{B}}$. Let us begin to construct a strategy $\sigma$ for player I in the game $\Gamma$. Define $\sigma(\emptyset) = \sigma'(\emptyset) = \{U_0', x_0\}$. Suppose that $n-1$ steps have been already done. If player II takes an open set $V_n$ with $x_n \in V_n \subset U_n'$ then one can take an open set $V_n' \subset V_n$ such that $x_n \in V_n'$ and $V_n' \in \mathcal{B}$. But then player I may follow the strategy $\sigma'$ and the induction goes on. $\square$

**Lemma 3.** *Let $D^*$ be a countable space $D \cup \{*\}$ with the only nonisolated point $*$. Assume that there is a decomposition $D = \bigsqcup_{n\in\mathbb{N}} D_n$ into a countable family of disjoint infinite subsets $D_n$ with $* \notin \mathrm{Cl}(D_n)$ for every $n$, and such that, for any neighborhood $U$ of $*$, some intersection $U \cap D_n$ is infinite.*

*Then the space $\mathcal{F}$ can be closely embedded into $C_p(D^*)$.*

**Proof.** By Lemma 1, we can identify $C_p(D^*)$ with $C_p^0(D^*)$. Let us also identify $D$ with the countable set $\{(n,k)\colon n = 1,2,\ldots \text{ and } k = 0,1,2,\ldots\}$. Denote by $D_k = \{(n,k)\colon n = 1,2,\ldots\}$ the horizontal 'layer' for every $k = 0,1,2,\ldots$, and let $E = \{(n,k)\colon n \geqslant 1 \text{ and } k \geqslant 1\}$. Define also the mapping $f\colon \mathbb{R}^{D_0} \to \mathbb{R}^E$ by the following formula:

$$f(x)(n,k) = \begin{cases} |x_n| & \text{if } k = 1, \\ |x_n|\left(\sum_{i_1<\cdots<i_{k-1}\leqslant n} |x_{i_1}x_{i_2}\cdots x_{i_{k-1}}|\right) & \text{if } 2 \leqslant k \leqslant n, \\ 0 & \text{if } k > n. \end{cases}$$

The value $f(x)(n,k)$ depends on the finite collection of values $x_i$ for $i < n$, therefore, by the definition of the pointwise topology, the map $f$ is continuous. If $x_n \neq 0$ and the cardinality of the set $\{i = 1,\ldots,n\colon x_i \neq 0\}$ is equal to $k$, then $f(x)(n,j)$ is not equal to 0 for $j = 1,\ldots,k$, but $f(x)(n,j) = 0$ for $j > k$. It means that for $x$ in $\mathcal{F}$ with $|\operatorname{supp} x| = k$ the height of the support of the point $f(x)$ is not greater than $k$, i.e., $f(x)(n,j) = 0$ for $j > k$ and all $n$.

We also define the mapping $g\colon \mathbb{R}^E \to \mathbb{R}^E$ by the formula

$$g(y)(n,k) = n\sum_{i=1}^{n} |y_{ik}|.$$

The map $g$ is continuous by the same reason as the above map $f$. Our multiplication by $n$ of sum in the definition of $g(y)$ implies the following assertion

$$\text{If } y(n_0, k) \neq 0 \text{ for some } n_0, \text{ then } \lim_{n\to\infty} g(y)(n,k) = \infty. \qquad (*)$$

Let $h = g \circ f : \mathbb{R}^{D_0} \to \mathbb{R}^E$. Being a continuous mapping $h$ has a closed graph $\operatorname{Gr} h$. The space $\operatorname{Gr} h$ lies in $\mathbb{R}^{D_0} \times \mathbb{R}^E \equiv \mathbb{R}^{D_0 \cup E} \equiv \mathbb{R}^D$ and is homeomorphic to $\mathbb{R}^{D_0}$ (it is the domain of $h$). Every point $\alpha \in \operatorname{Gr} h$ one can consider as a function $\alpha : D_0 \cup E \to \mathbb{R}$. Denote by $\widetilde{\alpha} : D_0 \cup E \cup \{*\} \to \mathbb{R}$ the trivial extension defined by the rule $\widetilde{\alpha}(*) = 0$. It gives us the homeomorphism $u : \mathbb{R}^{D_0} \to \mathbb{R}^{D^*}$, where $u(x) = (x, \widetilde{\alpha}(x))$. It is easy to see that the following assertion completes the proof.

$$u(x) \in C_p(D^*) \text{ if and only if } x \in \mathcal{F}. \qquad (**)$$

Let $x \in \mathcal{F}$. Then the height $\max\{l;\ f(x)(n,l) \neq \emptyset \text{ for some } n\}$ is not greater than $k$, where $k = |\operatorname{supp} x|$. Further, the value $g(y)(n,i)$ depends on the values of $y_{ji}$ of the same height $i$. Consequently, a support of $f(x) = g(h(x))$ completely covered by the union $\bigcup_{i=0}^k D_i$. Since $\widetilde{\alpha}(x) = 0$, then the point $*$ is not in $\operatorname{Cl}(\bigcup_{i=0}^k D_i)$ by conditions of lemma, and we get $u(x) \in C_p(D^*)$.

On the other hand, let $x$ not be in $\mathcal{F}$. It follows that $\operatorname{supp} x$ is infinite and the limit of $h(x)$ is equal to $\infty$ on every horizontal level. By the assumption of the lemma, there is a level with an infinite intersection with every neighbourhood of the point $*$, hence the function $u(x)$ is not continuous by the property $(*)$. □

**Theorem.** *For an ultrafilter $\phi \in \mathbb{N}^*$, the following are equivalent:*

(1) *the ultrafilter $\phi$ is not a $P$-point in $\mathbb{N}^*$,*

(2) *the space $\mathcal{F}$ of all points in $\mathbb{R}^{\mathbb{N}}$ with a finite support can be closely embedded into $C_p(\mathbb{N}_\phi)$,*

(3) *the space $\mathbb{Q}$ of all rational numbers can be closely embedded into $C_p(\mathbb{N}_\phi)$,*

(4) *$C_p(\mathbb{N}_\phi)$ is not a hereditary Baire space,*

(5) *player* I *has a winning strategy in the game $\Gamma$ (in $C_p(\mathbb{N}_\phi)$),*

(6) *player* I *has a winning strategy in the game $\Delta$ (in $\mathbb{N}_\phi$).*

**Proof.** Since the space $C_p(\mathbb{N}_\phi)$ is a separable metrizable space, equivalences (3) $\Leftrightarrow$ (4) and (4) $\Leftrightarrow$ (5) were proved by Hurewicz [3] (see also [1,2] for generalizations) and Debs [2, Theorem 4.1(b)], respectively.

The space $\mathbb{Q}$ can be identified with the closed subspace of $\mathcal{F}$ consisting of all $\{0,1\}$-valued sequences, therefore, (2) $\Rightarrow$ (3).

The implication (1) $\Rightarrow$ (2) is an easy consequence of Lemma 3.

So, we need to prove the implications (5) $\Rightarrow$ (6) $\Rightarrow$ (1). In both cases we will identify $C_p(\mathbb{N}_\phi)$ with $C_p^0(\mathbb{N}_\phi)$ by Lemma 1.

(5) $\Rightarrow$ (6) Let $\sigma$ be a winning strategy for player I in the game $\Gamma$ (which is played in $C_p(X)$) and suppose that there is no winning strategy for player I in the game $\Delta$ on space $X$. By Lemma 2 we can suppose that both players take their moves from the standard base of the space $C_p(X)$ consisting from sets of the form: $\bigcap_{i=1}^n \langle x_i, B_i \rangle$,

where $\langle x_i, B_i\rangle = \{f \in C_p(X);\ f(x_i) \in B_i\}$ and $B_i$ is an interval of reals for every $i = 1, \dots, n$.

Let $\sigma$ be a winning strategy for player I in the game $\Gamma_{\mathcal{B}}$ (that takes place in $C_p(\mathbb{N}_\phi)$). We are going to find a winning strategy $\tau_\sigma$ for player I in the game $\Delta$ (in the space $X$). If $\sigma(\emptyset) = (U_0, f_0)$ is the beginning move in $\Gamma_{\mathcal{B}}$, then let $\tau_\sigma(\emptyset) = S_0 = \{n \in \mathbb{N}\colon |f_0(n)| \geqslant 1\}$. It is clear that $S_0 \notin \phi$.

Suppose that the strategy $\tau_\sigma$ has been already constructed up to the $n$th move and let $U_n = \bigcap_{i=1}^k \langle x_i, (\alpha_i^n, \beta_i^n)\rangle$ with a function $f_n$ being the last move of player I in the game $\Gamma_{\mathcal{B}}$, where $(\alpha_i^n, \beta_i^n)$ is an interval of reals for every $i$.

Define an open subset $V_n$ in $U_n$ by the following conditions (a)–(d), and take it as the $n$th move of player II in the game $\Gamma_{\mathcal{B}}$.

(a) $V_n = \bigcap_{j=1}^l \langle x_j, (\gamma_j^n, \delta_j^n)\rangle$,

(b) $\operatorname{supp} V_n = T_n \cup \operatorname{supp} U_n$,

(c) if $x_j \in \operatorname{supp} U_n$, then $\alpha_j^n < \gamma_j^n < f_n(x_j) < \delta_j^n < \beta_j^n$ and $|\gamma_j^n - \delta_j^n| < |\beta_j^n - \alpha_j^n|/2$,

(d) if $x_j \in T_n$, then $(\gamma_j^n, \delta_j^n) = (-1/n, 1/n)$.

Further define

$$(U_{n+1}, f_{n+1}) = \sigma\big((U_0, f_0), V_0, \dots, (U_n, f_n), V_n\big)$$

and

$$\begin{aligned}\Delta(S_0, T_0, \dots, S_n, T_n) &= S_{n+1}\\ &= \{k \in \mathbb{N};\ |f_{n+1}(k)| \geqslant 1/(n+2)\}\setminus\bigcup_{i=1}^{n}(S_i \cup T_i).\end{aligned}$$

In fact, the last formula defines both sets $\Delta(S_0, T_0, \dots, S_n, T_n)$ and $S_{n+1}$. All needed sets are found and this finishes our inductive construction of strategy $\tau_\sigma$. Suppose that it is not a winning one and, therefore, the union $\bigcup_n T_n$ is in $\phi$. Condition (c) implies the existence of a single common point $f(t)$ of decreasing sequence of segments $[\alpha_t^n, \beta_t^n]$ for every $t \in \bigcup_n T_n$. Define $f(t) = 1$ for $t \notin \bigcup_n T_n$. By condition (d), $f(t) \to 0$ when $t \to \phi$, hence, $f$ is a continuous function and $\bigcap_n U_n \neq \emptyset$. Therefore, player I lost the corresponding play in the game $\Gamma_{\mathcal{B}}$ which is a contradiction.

(6) $\Rightarrow$ (1) Let $\sigma$ be a winning strategy for player I in the game $\Delta$. Suppose that assertion (1) is not true and $\phi$ is a $P$-point in $\mathbb{N}^*$. Therefore, if $\mathcal{A}$ is a countable family of subsets of $\mathbb{N}$ which is disjoint with $\phi$ then there exists a set $\Phi = \Phi(\mathcal{A})$ in $\phi$ with a finite intersection $\Phi \cap A$ for every $A \in \mathcal{A}$.

Define $\mathcal{A}$ be a family of all moves of player I in the game $\Delta$ under the strategy $\sigma$. The family $\mathcal{A}$ is countable since each move in every play of the game $\Delta$ depends on a choice of a finite set $T_n$.

We can assume that $\sigma(\emptyset) \cap \Phi = \emptyset$, otherwise one can take $\Phi\setminus\sigma(\emptyset)$ instead of $\Phi$ (recall that the set $\Phi \cap \sigma(\emptyset)$ is finite, hence, $\Phi\setminus\sigma(\emptyset) \in \phi$). We enumerate the set $\Phi$ as an increasing sequence $l_1, l_2, \dots$.

Now let us consider two plays $(S_0, T_0, S_1, T_1, \dots)$ and $(S_0', T_0', S_1', T_1', \dots)$ with moves of one play exactly after another. Let $S_0 = S_0' = \sigma(\emptyset)$, $T_0 = \emptyset$, $S_1 = \sigma(S_0, T_0)$ and $k_0 = 0$. Let us assume that the sets $S_n$ and $S_n'$ with a natural number $k_n$ were already constructed. Define $k'_n$ to be the first natural which is greater than three numbers: $k_n + 1$, $\max(\Phi \cap S'_n)$ and $l_{2n-1}$. Let $T_n = [k_n + 1, k'_n]$, $S_{n+1} = \sigma(S_0, T_0, \ldots, S_n, T_n)$ and $k_{n+1}$ be the supremum of numbers $k'_n + 1$, $\max(\Phi \cap S_{n+1})$ and $l_{2n}$. Further let $T'_n = [k'_n + 1, k_{n+1}]$. This finishes the inductive definitions.

By construction we get $\Phi \subset \bigcup_{n\in\mathbb{N}}(T_n \cup T'_n)$. Since $\Phi$ is in $\phi$ and the last family is an ultrafilter, then either $\bigcup_{n\in\mathbb{N}} T_n$ or $\bigcup_{n\in\mathbb{N}} T'_n$ is in $\phi$. Therefore, player II won in one of the above plays. It is a contradiction since both plays were played according to the same winning strategy $\sigma$ for player I in the game $\Delta$. $\square$

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